PROJECTIVE COVERS AS SUBQUOTIENTS OF ENLARGEMENTS

BY

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ABSTRACT

We show that not only completions of systems in various senses but also projective covers in the category of compact Hausdorff spaces may be obtained as subquotients of enlargements.

1. Introduction

It is known that enlargements of models often contain as subquotients, extensions of the models that are of importance classically. For example, as in [7], distributions may be regarded as equivalence classes of internal functions. As another example, in [5] and [6] completions of uniform spaces are studied from this point of view. We shall see that not only completions in the topological sense but also projective covers can be obtained from enlargements.

For background in non-standard analysis the reader is referred to [5], [6], or [7]. All enlargements considered will be higher order non-standard models.

2. Survey of various examples

We begin by rapidly surveying results which are essentially known.

The Stone-Čech compactification of a completely regular space X may be obtained as follows: Let X^* be an enlargement of X and let F be the class of bounded continuous functions on X. We define an equivalence relation on X^*:

$$x \sim y \text{ iff } {}^0f(x) = {}^0f(y) \text{ for all } f \in F.$$

Every $f \in F$ induces a function $\bar{f}$ on the equivalence classes, namely, $\bar{f}(\bar{x}) = {}^0f(x)$. Then X^* with the weak topology induced by the $\bar{f}$ is the Stone-Čech compactification of X.

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The second conjugate space of a Banach space B can be obtained in an analogous manner. Let $\gamma B = (x \in B^*: f(x)$ is finite for all $f \in B')$. Let $I = (x \in B^*: {}^0f(x) = 0$ for all $f \in B')$. Then $\gamma B/I$ is the second conjugate space of B.

Rings of quotients of rings of continuous functions can also be obtained in this manner. Let $R(X)$ be the ring of real valued continuous functions on the compact Hausdorff space X. Then it is easily seen using the method of enlargements that *every* (not necessarily continuous) function from X to R extends to a * continuous function from X^* to R^*. In particular, every continuous function on a dense open set of X into R can be extended to such a function. Hence by [1], the ring of quotients of $R(X)$ may be regarded as a subquotient of $[R(X)]^*$. Similarly, using [4] we can deal with injective hulls of C^* algebras.

Although the situation looks different, the ring of quotients of the integers J can also be obtained from an enlargement. In fact, if P is an infinite prime in J^*, then $J^*/(P)$ is a field containing the rationals.

Finally, we consider the case of Boolean algebras. Let B be an infinite Boolean algebra and B^* an enlargement. For $x \in B^*$ define $U(x) = (y \in B: y \geqq x)$ and $L(x) = (y \in B: y \leqq x)$. Define: $x \in \gamma B$ if

$$\bigcap_{\substack{y \in U(x) \\ z \in L(x)}} (y - z) = 0$$

where $\cap$ is understood to be in B.

Define $I = (x \in \gamma B: L(x) = \{0\})$. Then γB is a subalgebra of B^*, I is an ideal in γB and $\gamma B/I$ is the completion of B. Several natural questions with respect to γB and I have interesting answers. First, $I \neq \{0\}$. Secondly $B^* = \gamma B$ iff B is atomic. At the other extreme, if B is atomless, than $(\exists x \in B^*)\ [U(x) = \{1\}$ and $L(x) = \{0\}]$. The details are expected to appear in the proceedings of a symposium on nonstandard analysis held in Victoria in May 1972.

3. Projective covers

For background we refer the reader to [3]. The main example we consider is that of the projective cover P of a compact Hausdorff topological space X with enlargement X^*. This differs from all previous examples because P is a cover rather than an extension; in fact, it is rare for P to contain a copy of X. In addition, since all points of X^* are near-standard, it does not seem that X^* has much new to offer. Yet, even here, P can be obtained as a subquotient of X^*. The difference is that here the action will take place in $X^* - X$ for most of the usual spaces. The equivalence relation will thus be such that $X^* - X$ will have new things to offer after all.

We remind the reader that P is the Stone space of the Boolean algebra of regular open subsets of X.

To avoid cluttering up notation in the sequel, we shall use the same symbol A for a subset of X and the corresponding subset of X^*. The only possible danger of confusion is with respect to quantification since the other operations used commute with the correspondence. We therefore emphasize that all quantification considered will be understood to be in X, e.g. when the phrase "for all open sets U" is used, it will mean "for all sets of the form U^* where U is open in X". (As is well known, this usually is *not* the same as "for all * open sets in X^*".)

THEOREM 3.1. *The following conditions on a point $x \in X^*$ are equivalent:*

1) *x is not in the closure of two disjoint open sets*

2) *The class of regular open sets containing x is an ultrafilter in the Boolean algebra of regular open sets.*

PROOF. (1) $\Rightarrow$ (2). Since inclusion and intersection agree with the usual set theoretic meaning, the class containing x is always a filter. (It is worthwhile to note that this fails if regular closed sets are used.) Hence to prove (2) we need show only that for an arbitrary regular open set U, $x \in U$ or $x \in \mathrm{int}(U')$. Otherwise $x \in U' = \overline{\mathrm{int}(U')}$ and $x \in [\mathrm{int}(U')]' = \bar{U}$. Since U and $\mathrm{int}(U')$ are complementary sets, this contradicts (1).

(2) $\Rightarrow$ (1). Suppose U and V are disjoint open sets. Then $\mathrm{ext}(U)$ and ext.ext.U are regular open sets containing V and U respectively. Furthermore, ext.ext.U is the Boolean algebra complement of ext.U. Hence by (2), $x \in \mathrm{ext}.(U)$ or $x \in \mathrm{ext.ext.}(U)$, hence $x \notin \bar{U}$ or $x \notin \bar{V}$. This proves (1). Q.E.D.

Note that, as the theorem is stated, it cannot be proved by transfer. However, by phrasing the theorem in terms of specific U and V, this can be done. The latter is a possible alternative in style for readers who prefer not to work directly in X^*.

Let γX be the set of all x satisfying either condition in Theorem 3.1. It is an easy exercise that $X \subset \gamma X$ iff X is extremely disconnected. Another exercise of interest is that for metric spaces $X \cap \gamma X$ consists of the isolated points of X. Hence any dense-in-itself metric space satisfies $\gamma X \subset X^* - X$.

Let $F(x)$ be the ultrafilter of regular open sets containing x. We now define an equivalence relation in γX: $x \sim y$ if $F(x) = F(y)$. Let the quotient set be δX. As usual, the class containing x is denoted by $\bar{x}$.

Since X^* is an enlargement of X, it follows by the usual method that every ultrafilter F has a non-empty intersection in X^*. For any point x in the intersection, $F = F(x)$. (From the point of view of regular closed sets, there is even a unique point in X which is in the intersection. However, different ultrafilters may lead to the same point. In fact, this is the usual map $P \to X$. By considering regular *open* sets and intersections in X^*, distinct ultrafilters give rise to distinct equivalence classes of points. This is what will permit us to replace P by δX.)

We use the one-one correspondence between δX and P to obtain a topology on δX. Alternatively, the topology may be obtained as the quotient of the relative topology where the regular open sets are taken as a basis for the open sets of X^*. The projective cover can then be regarded as the map $\gamma X/\sim \xrightarrow{e} X$ which sends the equivalence class containing x into ${}^0x$. This corresponds to the usual map $P \to X$ since ${}^0x$ necessarily lies in the closure of each set in $F(x)$. We now give a non-standard proof that $\delta X \xrightarrow{e} X$ is an essential cover. In more detail we have:

THEOREM 3.2. *The map $\delta X \xrightarrow{e} X$ where $e(\bar{x}) = {}^0x$ is well-defined continuous and onto but is not onto when restricted to a proper closed subset of X.*

REMARK. We have no substitute for the known argument that $\delta X = P$ is projective.

NOTE. In contrast to our previous convention, we now prefer to use an asterisk to distinguish sets in X from corresponding sets in X^*. This is because we are now dealing with both X^* and X simultaneously.

PROOF. Suppose ${}^0x \neq {}^0y$. Then there exist regular open sets U and V such that ${}^0x \in U$, ${}^0y \in V$ and $U \cap V = \phi$. Then $x \in U^*$ and $y \in V^*$. Hence $y \notin U^*$. Thus $x \nsim y$. This proves that the map is well-defined.

Suppose $e(\bar{x}) = {}^0x$ and U is open such that ${}^0x \in U$. Let ${}^0x \in V \subset \bar{V} \subset U$ where V is regular open. In addition $x \in V^*$. Now the quotient of V^* is an open set in δX containing $\bar{x}$. Any point $\bar{y}$ such that $y \in V^*$ satisfies $e(\bar{y}) = {}^0y \in \bar{V} \subset U$. This proves continuity.

Let $x \in X$. The collection of regular open sets containing x is a filter (in the Boolean algebra sense) and is hence contained in an ultrafilter F. Let $F = F(y)$. Then y is contained in U^* for every open set U in F. Since every open set containing x includes a regular open set containing x by regularity, y is in the monad of x. Hence $e(\bar{y}) = x$.

For the last part it suffices to consider complements of basis sets. Let U be a non-empty regular open set. Then $\{\bar{x}: x \notin U^*\}$ is a typical set C of this kind. Now ${}^0x \in U \Rightarrow x \in U^*$. Hence $x \notin U^* \Rightarrow e(\bar{x}) \notin U$. Thus $\delta(C) \cap U = \emptyset$. Q.E.D.

We have thus seen that by a suitable representation of the projective cover of X, the inverse image of a given point x consists of equivalence classes in its monad; x will lie in one of the equivalence classes if and only if $x \in \gamma X$.

We close with an important characterization of the points $x \in X$ which lie in γX.

THEOREM 3.3. *$e^{-1}(x)$ consists of a single point iff $x \in \gamma X$.*

In this case $e^{-1}(x)$ is precisely the single equivalence class consisting of the monad of x.

PROOF. Suppose $x \in \gamma X$. Since every point in the monad of x is contained in U^* for every open set containing x, it follows that every point y in the monad of x is contained in every set in $F(x)$. Hence $F(y) = F(x)$. Therefore $y \in \gamma X$ and all y in the monad of x are equivalent. Thus $e^{-1}(x)$ consists of a single point.

Now suppose $x \notin \gamma X$. Then the collection of regular open sets containing x can be extended in at least two ways to ultrafilters F_1 and F_2. Let $F_1 = F(y)$ and $F_2 = F(z)$. Then $\bar{y} \neq \bar{z}$ and $e(\bar{y}) = e(\bar{z}) = x$ as is easily seen by the same argument that showed that e is onto. Q.E.D.

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